SPECTRAL ELEMENT METHODS FOR THE SOLUTION OF BOUNDARY VALUE PROBLEM BY USING EXPANSION IN CHEBYSHEV POLYNOMIALS

M.N. Bashir [1] , M. O. Ahmad[2] , M. luqman[3] and M. A. Meraj[4]

1. Department of Mathematics, University of Engineering and Technology, lahore , Pakistan.
Department of Mathematics, COMSATS institute of information of Technology, sahiwal
2. Department of Mathematics, University of Lahore.pakistan
3. Department of Mathematics, University of Engineering and Technology, lahore , Pakistan.
4. Assistant professor Department of Mathematics, COMSATS institute of information of Technology, sahiwal

ABSTRACT: *Not all differential equations can be solved analytically, to overcome this problem, there is need to search for an accurate approximate solution. Approach: The objective of this study was to find an accurate approximation technique (scheme) for solving linear differential equations. By exploiting the Trigonometric identity property of the Chebyshev polynomial, we developed a numerical scheme referred to as the spectral Galerkin method. Results: With the scheme developed, we were able to obtain approximate solution for certain linear differential equations. Conclusion: The numerical scheme developed in this study competes favorably with solutions obtained with standard and well known spectral methods. We presented numerical examples to validate our results and claim.*

Key Words: Chebyshev polynomial, linear ordinary differential equations, Spectral method, Spectral tau method Pseudospectral method, spectral Galerkin method.

INTRODUCTION

The fundamental problem of approximation of a function by interpolation on an interval paved way for the spectral methods which are found to be successful for the numerical solution of ordinary and partial differential equations. Spectral representations of analytic studies of differential equations have been in used since the days of Fourier. Their application to Numerical solution of ordinary differential equations refers, at least to the time of Lanczos[10]. Summary of survey of some applications is given in[8]. Some present spectral methods can also be traced back to the "'method of weighted residuals"' of Finlayson and Scriven[6].Spectral methods can be viewed as an extreme development of the class of discretization scheme for differential equations known as the Method of Weighted Residuals (MWR)[6]. In MWR, the use of approximating functions (called trial functions) is central. These functions are used as basis functions for a truncated series expansion of the solution. Another function called the test functions (also known as the weight functions) are used to ensure that the differential equation is satisfied as close as possible by the truncated series expansion. Among the spectral schemes the three most commonly used are the Tau, Galerkin and collocation (also called pseudo-spectral) methods. What distinguishes between these methods is the choice of the test functions employed. Galerkin and Tau method are implemented in terms of the expansion coefficients[5], whereas collocation methods are implemented in terms of physical space values of unknown function. Over the past two decades, spectral methods with their current forms appeared as attractive ways in most applications. Some more details on spectral methods could be seen in[9,11-13]. The basic idea of spectral methods to solve differential equations is to expand the solution function as a finite series of very smooth basis functions, as given below:

$$y = \sum_{i=0}^{N} a_i T_i(x)$$

where, T_i represents Chebyshev or Legendre polynomials[14] (for more on Chebyshev polynomials).

If $y \in C^{\infty}[a,b]$the error produced by the approximation approaches zero with exponential rate[4] as N becomes too large (tends to infinity). This phenomenon is referred to as 'spectral accuracy [8]. The accuracy of the derivative obtained by direct term-by-term differentiation of such truncated expansion naturally deteriorates [4], but for low-order derivatives and sufficiently high-order truncations this deterioration is negligible, compared to the restrictions in accuracy introduced by typical difference approximations. In [2] and[3], the researchers focused on differential equations in which one of the coefficient function or solution function is not analytic on the interval of definition. Weak aspect of spectral methods in solving this kind of problems were studied in[2] and[3] and the researchers came up with modifications to the spectral method which proved to be more efficient when compared with existing ones. In this article, we present a variation of the spectral Galerkin method to solve the problems The spectral Galerkin method (the method introduced in this article) is seen to be efficient and competes favorable with other wellknown standard methods like the Tau method, and the Pseudo-spectral (collocation) method.

DIFFERENTIAL EQUATION SOLVERS

The Weighted Residual Method

Mathematically, following system can be considered:

$$Lu(x) = S(x) \quad for \quad x \in u \tag{1}$$

$$u(y) = 0 \quad for \quad y \in \partial u \tag{2}$$

Where

L & S = linear differential operators.

A function 'u' is considered permissible solution of this system, if Eq. (2) is satisfied exactly and if residual ($R \equiv S - Lu$) becomes small. In order to quantify what the meaning of this "small" is, the weighted residual method depends on (N + 1) tests functions ξ_n and scalar product of R with these functions becomes zero:

$$(\xi_k, R) = 0, \quad \forall k \le N \quad (3)$$

This is clear that with the increase of N the solution becomes more close to the exact solution . Different types of spectral solvers can be generated by selection of spectral basis and test function. Now we will present the three most popular methods in the following and we will apply them for very simple case,

A Test Problem

A TEST PROBLEM

We suggest for the solution of the equation:

$$u''(x) - 4u'(x) + 4u(x) = f(x) \quad (4)$$

with $x \in [-1,1]$ and $f(x) = \exp(x) - 4e/(1+e^2)$. As boundary conditions, it is simply asked that at the boundaries, the solution becomes zero "

$$u(-1) = 0 \text{ and } u(1) = 0 \quad (5)$$

The solution is unique under those conditions

$$u_{sol}(x) = \exp(x) - \frac{\sinh(1)}{\sinh(2)} \exp(2x) - \frac{e}{1+e^2} \quad (6)$$

Let's consider that the solution of this equation is not a polynomial.

Linear operator will be:

$$L = \frac{d^2}{dx^2} - 4\frac{d}{dx} + 4Id .$$

Using the elementary linear operations.

One can build the matrix representation of L, which will be useful in the implementation of the different solvers.

Let $u = \sum_{i=0}^{N} a_i T_i(x)$

Then

$$Lu = \sum_{i=0}^{N}\sum_{j=0}^{N} L_{ij} a_i T_i(x) .$$

For this case, and for N equal to four (4),

$$L_{ij} = \begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 0 & 0 & 0 & 4 & -32 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \quad (7)$$

The Tau-Method

In this method, the test functions ξ_n are selected to be identical as the spectral functions of decomposition. Let us use the Chebyshev polynomials T_i. The residual equations (3) are then:

$$(T_n, Lu - S) = 0 \ \forall n \le N \quad (8)$$

By using the definition of the matrix L_{ij}, these equations can be written as

$$\sum_{i=0}^{N} L_{nj} a_j = \tilde{s}_n \quad \forall n \le N \quad (9)$$

where the $\tilde{s}_n$ are the spectral coefficients of the source S. However, due to the presence of homogeneous solutions of L, this set of N + 1 equations are degenerated and by imposing the boundary conditions before solution. In the Tau-method, the boundary conditions are imposed as extra equations. In our special case these can be written:

$$\left.\begin{array}{l} u(x=-1) = 0 \Rightarrow \sum_{j=0}^{N} (-1)^j a_j = 0 \\ u(x=+1) = 0 \Rightarrow \sum_{j=0}^{N} a_j = 0 \end{array}\right\} \quad (10)$$

Last two residual equations is relaxed and replaced by two boundary conditions to find an invertible system having a_n as unknowns. Relaxing the last two equations is not a problem. Indeed, if the function is regular, the coefficients arc quickly lessening and so the solution should come close to the exact solution.

In this example, and for *N equal to four* the matrix format of equation is

$$\begin{pmatrix} 4 & -4 & 4 & -12 & 32 \\ 0 & 4 & -16 & 24 & -32 \\ 0 & 0 & 4 & -24 & 48 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -0.03 \\ 1.13 \\ 0.27 \\ 0 \\ 0 \end{pmatrix} \quad (11)$$

The solution is :

a_0 = 0.1453773585

a_1 = 0.07863207547

a_2 = –0.1218396226

a_3 = –0.07863207547

a_4 = –0.02353773585

Table (1) shows the comparison of numerical solution and exact solution.

Table – 1: Numerical Solution Obtained

Points x_i	Exact Solution	Numerical Solution	Error
$x_0 = -1$	0	0	0
$x_1 = -0.75$	0.07603918898	0.04974498823	0.02629420075
$x_2 = -0.5$	0.1633006009	0.1001179245	0.0631825164
$x_3 = -0.25$	0.2582412532	0.1657650354	0.0924762178
$x_4 = 0.0$	0.3519457263	0.2436792453	0.108266481
$x_5 = 0.25$	0.4257678471	0.3132001769	0.1125676702
$x_6 = 0.5$	0.4438970559	0.3360141509	0.107882905
$x_7 = 0.75$	0.3407840021	0.2561541863	0.0846298158
$x_8 = 1.00$	0	0	0

Fig. (1) shows the comparison of numerical solution and exact solution.

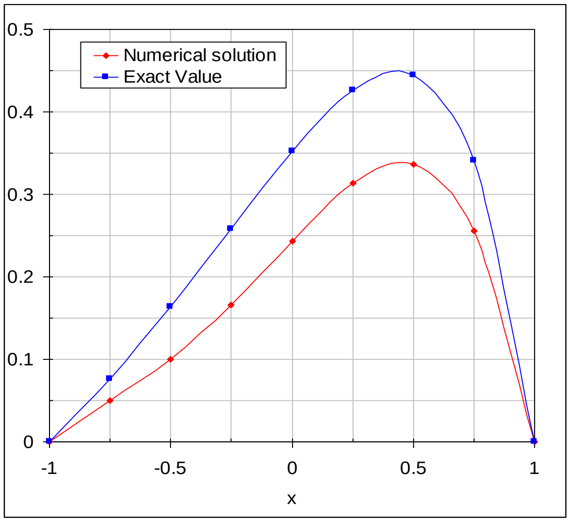

Fig. 1 The Tau Method for N = 4

PSEUDOSPECTRAL METHOD

We will use the spectral approximation for N = 4

$$U = \sum_{i=0}^{4} a_i T_i$$

$U = a_0T_0 - a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4$ .... (12)

$\Rightarrow u(-1) = a_0 - a_1 + a_2 - a_3 + a_4 = 0$ .... (13)

$\Rightarrow u(1) = a_{0} + a_1 + a_2 + a_3 + a_4 = 0$ .... (14)

$u' = a_1T_0 + a_2(4T_1) + a_3(6T_2 + 3T_0) + a_4(8T_3 + 8T_1)$

$u'' = 4a_2T_0 + 24a_3T_1 + a_4(48T_2 + 32T_0)$

Given equation becomes

$4a_0T_0 + 4a_1(T_1 - T_0) + 4a_2[2T_0 - 4T_1 + T_2] + 4a_3[-3T_0 + 6T_1 - 6T_2 + T_3] + 4a_4[8T_0 + 8T_1 + 12T_2 - 8T_3 + T_4] = e^x - \dfrac{4e}{1+e^2}$

$x = -\dfrac{1}{\sqrt{2}} \Rightarrow$

$4a_0 - (4+2\sqrt{2})a_1 + (4+8\sqrt{2})a_2 - (12+10\sqrt{2})a_3 + 28a_4 = -0.8030$

$x = 0 \Rightarrow 4a_0 - 4a_1 + 0a_2 + 12a_3 - 12a_4 = -0.2961$

$x = \dfrac{1}{\sqrt{2}} \Rightarrow$

$4a_0 - (4-2\sqrt{2})a_1 + (4-8\sqrt{2})a_2 - (12-10\sqrt{2})a_3 + 28a_4 = -0.7320$

The matrix form for the pseudospectral method is: (When N equal to four)

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 4 & -(4+2\sqrt{2}) & 4+8\sqrt{2} & -(12+10\sqrt{2}) & 28 \\ 4 & -4 & 0 & 12 & -12 \\ 4 & -(4-2\sqrt{2}) & 4-8\sqrt{2} & -(12-10\sqrt{2}) & 28 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.8030 \\ -0.2961 \\ 0.7320 \\ 0 \end{bmatrix}$$

The solution is :

$a_0 = 0.1875410023$

$a_1 = 0.08866783409$

$a_2 = -0.1565058909$

$a_3 = -0.08866783409$

$a_4 = -0.03103511136$

Table (2) shows the comparison of numerical solution and exact solution

Table – 2: Numerical Solution Obtained

Points x_i	Exact Solution	Numerical Solution	Error
$x_0 = -1$	0	0	0
$x_1 = -0.75$	0.07603918898	0.08166649782	-5.62730884×10^{-3}
$x_2 = -0.50$	0.1633006009	0.1483097523	0.0149908486
$x_3 = -0.25$	0.2582412532	0.2248701595	0.0333710937
$x_4 = 0.0$	0.3519457263	0.3130117818	0.0389339445
$x_5 = 0.25$	0.4257678471	0.3911223484	0.0346454987
$x_6 = 0.50$	0.4438970559	0.4143132546	0.0295838013
$x_7 = 0.75$	0.3407840021	0.3144195623	0.0263644398
$x_8 = 1.00$	0	0	0

Fig. (2). shows the comparison of numerical solution and exact solution

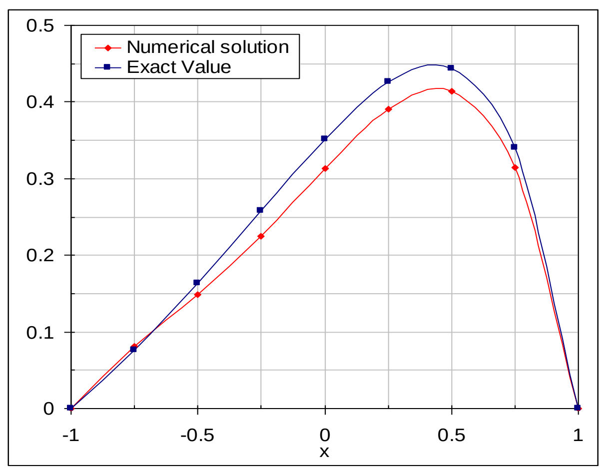

Fig. 2. Pseudospectral Method for N = 4

THE SPECTRAL GALERKIN METHOD (N = 4)

We will use the Spectral approximation for N = 4.

Let $u = \sum_{i=0}^{4} a_i T_i$

$$= a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4 \qquad (16)$$

The basic idea of the Galerkin method is to expand the solution, not in terms of usual orthogonal polynomials, but on some linear combinations of polynomials that fulfill the boundary conditions. One then talks of Galerkin basis. The particular choice of basis is of course important and it is rather hard to give a general recipe. However, it is usually better if the Galerkin basis can be easily written in terms of the original basis.

For the boundary conditions of our example, it is easy to see that the following choice of Galerkin basis φ_i is a valid one:

- $\varphi_{2k}(x) = T_{2k+2}(x) - T_0(x)$
- $\varphi_{2k+1}(x) = T_{2k+3}(x) - T_1(x)$

Note that φ_i's are not orthogonal polynomials

$$u = \sum_{i=0}^{2} b_i \varphi_i = \sum_{i=0}^{4} a_i T_i$$

$u = b_0 (2x^2 - 2) + b_1 (4x^3 - 4x) + b_2 (8x^4 - 8x^2)$

$u = - T_0 (b_0 - b_2) - b_1T_1 + b_0T_2 + b_1T_3 + b_2T_4$

Differentiate with respect to x in equation (17)

$u' = 2b_1T_0 + 4T_1 (b_0 + 2b_2) + 6b_1T_2 + 8b_2T_3$

Differentiate with respect to x in equation (19)

$u'' = 4T_0 (b_0 + 8b_2) + 24b_1T_1 + 48b_2T_2$

Then given equation becomes

$$4[-2b_1 + 7b_2]\, T_0 + 4\, [-4b_0 + 5b_1 - 8b_2]\, T_1 + 4[b_0 - 6b_1 + 12b_2]\, T_2 - 4\, [8b_2 - b_1]\, T_3 + 4b_2T_4 = e^x - \frac{4e}{1+e^2}$$

The Residual equations (3) are then:

$(\varphi_i, R) = 0 \qquad i = 0, 1, 2$

$\Rightarrow\ 2b_0 - 4\pi b_1 - 4\pi b_2 = 0.5208457121$ (20)

$\Rightarrow\ 8\pi b_0 - 8\pi b_1 + 0b_2 = -1.705855528$ (21)

$\Rightarrow\ 8\pi b_1 - 26\pi b_2 = 0.1029807468$ (22)

The matrix format of the Galerkin method is: (When N equal to four)

$$\begin{pmatrix} 2\pi & -4\pi & -4\pi \\ 8\pi & -8\pi & 0 \\ 0 & 8\pi & -26\pi \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0.5208457121 \\ -1.705855528 \\ 0.1029807468 \end{pmatrix}$$

The Galerkin coefficients are:

$b_0 = -0.1596460858$

$b_1 = -0.09177225092$

$b_2 = -0.02949837681$

and the standard ones:

$a_0 = 0.1891444626$

$a_1 = 0.09177225092$

$a_2 = -0.1596460858$

$a_3 = -0.09177225092$

$a_4 = -0.02949837681$

Table (3) shows the comparison of numerical solution and exact solution

Table – 3: Numerical solution Obtained

Points x_i	Exact Solution	Numerical Solution	Error
$x_0 = -1$	0	0	0
$x_1 = -0.75$	0.07603918898	0.07731417508	$-1.2749861\text{x}10^{-3}$
$x_2 = 0.5$	0.1633006009	0.1460583175	0.0172422834
$x_3 = -0.25$	0.2582412532	0.2271272898	0.0311139634
$x_4 = 0.0$	0.3519457263	0.3192921716	0.326535547
$x_5 = 0.25$	0.4257678471	0.3992002602	0.0265675869
$X_6 = 0.5$	0.4438970559	0.4213750703	0.0225219856
$x_7 = 0.75$	0.3407840021	0.3182163337	0.0225676684
$x_8 = 1$	0	0	0

Fig. (3). shows the comparison of numerical solution and exact solution

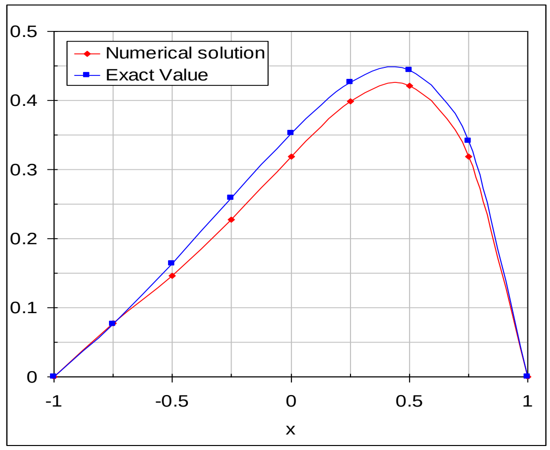

Fig. 3 The Galerkin Method for N=4

COMPARISON OF EXACT SOLUTION WITH TAU, PSUDOSPECTRAL AND GALERKIN

In the Figure (4) the blue line indicates the solution of linear second order boundary value problem while the red, green and black lines indicate the numerical solution obtained using Spectral Galerkin Method, Pseudospectral Method and Spectral Tau Method respectively. It is clear from graph; the results obtained by Spectral Galerkin Method are very close to exact solution. It is clearly seen that Galerkin's Method gives more accurate solution as compared to the Pseudospectral and Tau.

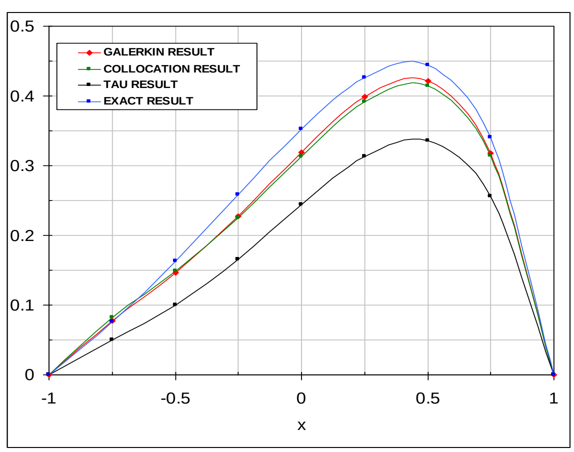

Fig. 4 Results for N = 4

SPECTRAL GALERKIN METHOD FOR N = 8

Now, solving the same problem taking $N = 8$

The matrix format for the Galerkin method is (When N equal to 8)

$$\begin{bmatrix} 2\pi & -4\pi & -4\pi & -4\pi & -8\pi & -4\pi & -12\pi \\ 8\pi & -8\pi & 0 & -18\pi & -26\pi & -26\pi & 0 \\ 0 & 8\pi & -26\pi & -4\pi & -4\pi & -4\pi & -60\pi \\ 8\pi & -10\pi & 16\pi & -56\pi & 0 & -82\pi & 0 \\ 0 & 8\pi & -28\pi & 16\pi & -10\pi & -4\pi & -140\pi \\ 8\pi & -10\pi & 16\pi & -58\pi & 24\pi & -164\pi & 0 \\ 0 & 8\pi & -28\pi & 16\pi & -104\pi & 24\pi & -250\pi \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = \begin{bmatrix} 0.5208 \\ -1.7058 \\ 0.1030 \\ -1.7746 \\ 0.0945 \\ -1.7755 \\ 0.0944 \end{bmatrix}$$

The Galerkin coefficients are

$b_0 = -0.1749792094$

$b_1 = -0.09352919879$

$b_2 = -0.02740017008$

$b_3 = -5.824766329 \times 10^{-3}$

$b_4 = -9.912187319 \times 10^{-4}$

$b_5 = -1.447543067 \times 10^{-4}$

$b_6 = -1.862085517 \times 10^{-5}$

and the standard ones

$a_0 = 0.2033892191$

$a_1 = 0.0994987194$

$a_2 = -0.1749792094$

$a_3 = -0.09352919879$

$a_4 = -0.02740017008$

$a_5 = -5.824766329 \times 10^{-3}$

$a_6 = -9.91218739 \times 10^{-4}$

$a_7 = -1.447543067 \times 10^{-4}$

$a_8 = -1.862085517 \times 10^{-5}$

Table (4). shows the comparison of numerical solution and exact solution

Table – 4: Numerical Solution Obtained

Points x_i	Solution	Exact Solution	Error
–1	0	0	0
–0.75	0.0760542845	0.07603918898	$-1.50955\text{x}10^{-5}$
–0.5	0.1633032024	0.1633006009	$-2.6015\text{x}10^{-6}$
–0.25	0.2582358038	0.2582412532	$5.4494\text{x}10^{-6}$
0.0	0.3519408563	0.3519457263	$4.87\text{x}10^{-6}$
0.25	0.4257682052	0.4257678471	$-3.581\text{x}10^{-7}$
0.5	0.4438907987	0.4438970559	$6.2572\text{x}10^{-6}$
0.75	0.3407810688	0.3407840021	$2.9333\text{x}10^{-6}$
1.00	0	0	0

Fig. (5) shows the comparison of numerical solution and exact solution

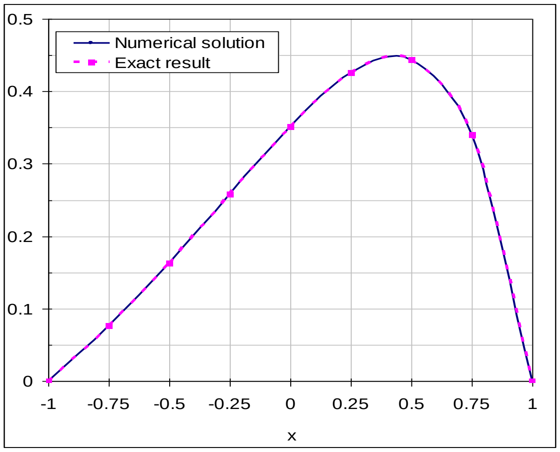

Fig. 5 The Galerkin Method for N=8

COMPARISON BETWEEN EXACT SOLUTION and GALERKIN METHOD (N equal to four & N equal to eight)

In the Figure (6) the pink doted line indicates the exact solution of linear second order boundary value problem while the blue and red lines indicate the numerical solution obtained using Spectral Galerkin Method for N = 8 and N = 4 respectively. It is clear from graph, the results obtained by spectral Galerkin Method for N = 8 are very close to exact solution. It is clearly seen that Galerkin Method for N = 8 gives more accurate solution as compared to Galerkin Method for N = 4.

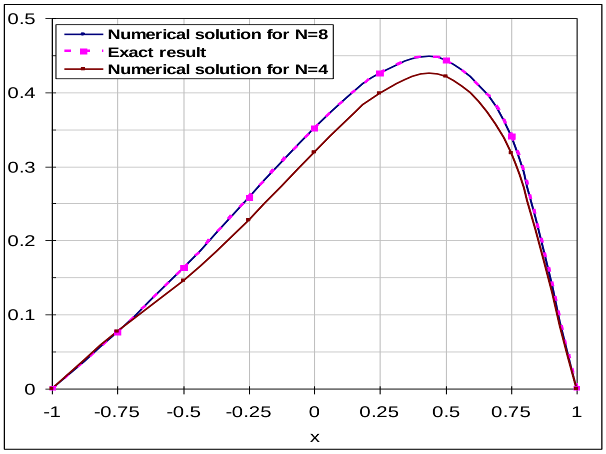

Fig. 6 Comparison of Results for N = 4 and N = 8

CONCLUSIONS

The Spectral Methods are used to solve linear second order boundary value problem for ordinary differential equation. We also studied the accuracy of the developed scheme. Problem is solved numerically using spectral Methods. The numerical results are then compared with the exact solution. It is observed that Spectral Galerkin Method gives more accurate solution as compared to Pseudospectral (Collocation) Tau.

It is also observed that accuracy of the results by using Spectral Galerkin Method can be improved by increasing the number of terms. We note that, the solution obtained by Spectral Galerkin Method is very close to exact solution for N = 8 as compared to results obtained for N = 4.

REFERENCES

[1] Ascher, U.M., R.M. Mahheij and R.d. Russell, 1988. Numerical Solution of Boundary Value Problem for Ordinary Differential Equations. 1st Edn.,Prentice-Hall Inc., USA., ISBN: 0-13-627266-5, pp: 619.

[2] Babolian, E. and M.M. Hosseini, 2002. A modified spectral method for numerical solution of ordinary differential equations with non-analytic solution. Applied Math. Comput., 132: 341-351. DOI: 10.1016/S0096-3003(01)00197-7

[3] Babolian, E., M. Bromilow, R. England and M. Savari,2007. A modification of pseudo-spectral method for solving a linear ODEs with singularity. Applied Math. Comput., 188: 1260-1266. DOI: 10.1016/j.amc.2006.10.079.

[4] Canuto, C., M. Hussaini, A. Quarteroni and T. Zang, 1988. Spectral Methods in Fluid Dynamics. Springer, Berlin, ISBN: 10: 0387173714, pp: 557.

[5] Delves, L.M. and J.L. Mohamed, 1985.Computational Methods for Integral Equations. 1st Edn., Cambridge University Press, Cambridge, ISBN: 10: 0521266297, pp: 388.

[6] Finlayson, A. and L.E. Scriven, 1996. The method of weighted residuals. Applied Mech. Rev., 19: 735-748.http://www.scribd.com/doc/8984439/Method-of-Weighted-Residuals

[7] Fornberg, B., 1996. A Practical Guide to Pseudo-Spectral Methods. Illustrated Edn., Cambridge University Press, Cambridge, ISBN: 0521645646, pp: 242.

[8] Gottlieb, D. and S. Orszag, 1977. Numerical Analysis of Spectral Methods, Theory and Applications. 6th Edn., SIAM, Philadephia, PA.,ISBN: 0898710235, pp: 172.

[9] E. Gourgoulhon, 2002. Introduction to Spectral Methods, 4th EU Network meeting, Palma de Mallorca, Sept. 2002.

[10] Lanczos, C., 1938. Trigonometric interpolation of empirical and analytic functions, J. Math. Phys., 17: 123-199.

[11] Ortiz, E.L., 1969. The tau method. SIAM. J. Numer. Anal., 6: 480-492. http://www.jstor.org/stable/2949509

[12] Ortiz, E.L. and J.H. Freilich, 1982. Numerical solution of system of ordinary differential equations with tau method: An error analysis.Math. Comput., 39: 467-479.http://www.jstor.org/stable/2007325

[13] Ortiz, E.L. and T. Chaves, 1968. On the numerical solution of two-point boundary value problems for linear differential equations ZAMM. J. Applied Math. Mech., 48: 415-418. DOI: 10.1002/zamm.19680480607

[14] Parker, I.B. and L. Fox, 1972. Chebyshev Polynomials in Numerical Analysis. 2nd Edn.,Oxford University Press, Oxford, ISBN: 13:9780198596141, pp: 216...

[15] Karageorghis, A., & Phillips, T.N., Spectral Collocation Methods for Stokes Flow in contraction geometries and unbounded domains, J Compute. Phys. 80(1989), pp. 3 14-330.

[16] Morchoisne Y., Inhomogeneous flow calculation by spectral methods, monodomain and mttltidomain techniques, in: Spectral Methods for Partial Dfferential Equations, Voigt, R.G., Gottlieb, D., & I-Jussaini, M.Y., Eds., SIAM, Philadelphia (1984), pp. 180-208.

[17] Patera, A.T., A spectral element method for fluid dynamics: laminar flow in a channel expansion J Compute. Phys., 54 (1984), pp. 468-488.

[18] Phillips, T. N., & Davies, A.R., On Semi-infinite S ectral Elements for Poisson ‘s Problem with Reentrant Boundary Singularities, J. Comput. Appl. Math, 21(1988), pp. 173-188.

[19] Phillips, T.N., & Karageorghis, A., Chebyshev Collocation Methods for for the solution of the incompressible Navier-Stokes equations in complex geometries, SIAM J. Sci. Stat. Comput., 10 (1989), pp. 89-103.

[20] Prenter, P.M., Splines and Variational Methods, Wiley, New York (1975). [34] Schatz, A., & Wahlbin, L.B., Maximum norm estimates in the finite element method on plane polygonal domains, Part 1, Math. Comp., 32 (1978), pp. 155-173.

[21] Shepley, L. Ross, “Differential Equation” 3rd ed., WSE, Willey.